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Research note

A new efficient method for solving the nonlinear Fokker–Planck equation

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Abstract In this paper, we propose a new technique for solving the nonlinear Fokker–Planck equation. This algorithm is based on the Laplace transform and new homotopy perturbation methods. The new technique is applied to solving two mathematical models of this problem. We show that the present approach is relatively easy, efficient and highly accurate.

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1. Introduction

In recent years, the increasing interest of scientists and engineers has been devoted to analytical asymptotic techniques for solving nonlinear problems, and many new numerical techniques have been widely applied to nonlinear problems. Based on homotopy, which is a basic concept in topology, a general analytical method, namely, the Homotopy Perturbation Method (HPM) was established by He [1–7] in 1998, to obtain a series of solutions of nonlinear differential equations. We apply a new version of the HPM that efficiently solves the Fokker–Planck equation. He's HPM has been already used by many mathematicians and engineers to solve various functional equations. In this method, the nonlinear problem is transferred to an infinite number of sub-problems and, then, the solution is approximated by the sum of the solutions of the first several sub-problems. This simple method has been applied to solve linear and nonlinear equations of heat transfer [8–10], fluid

mechanics [11], nonlinear Schrödinger equations [12], integral equations [13], boundary value problems [14], fractional KdV–Burgers equation [15] and the nonlinear system of second order boundary value problems [16].

The Fokker–Planck equation arises in a number of different fields in natural sciences [17], including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker–Planck Equation (FPE), first applied to investigate the Brownian motion of particles [18], is now largely employed in various generalized forms.

The general Fokker–Planck equation for variable x has the form [18]:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial^2 x} B(x) \right] u, \quad (1)$$

with the initial condition given by:

$$u(x, 0) = f(x), \quad x \in \mathbb{R}, \quad (2)$$

where $u(x, 0)$ is unknown. In Eq. (1), $B(x) > 0$ is called the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time, i.e:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial^2 x} B(x, t) \right] u. \quad (3)$$

Mathematically, this equation is a linear second order partial differential equation of the parabolic type. Roughly speaking, it

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is a diffusion equation with an additional first order derivative with respect to x .

The Nonlinear Fokker–Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing [19], and is written in the following form:

$$\frac{\partial u}{\partial t} = \left[- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i(X, t, u) + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{ij}(X, t, u) \right] u, \quad (4)$$

where $X = x_1, x_2, \dots, x_N$. Notice that when $A_i X, t, u = A_i X$ and $B_{ij} X, t, u = B_{ij} X$, the nonlinear Fokker–Planck equation (4) reduces to the linear Fokker–Planck equation. Because of the large number of applications of the Fokker–Planck equation, much work has been done to find the numerical solution of this equation [20–25].

Recently, Aminikhah and his co-associates proposed a new form of the Homotopy Perturbation Method (NHPM) to solve ordinary differential equations [26]. In their approach, the solution is considered as an infinite series that converges rapidly to exact solutions. Afterwards, they solved the system of ordinary differential equations [27], the nonlinear Blasius equation [28], the quadratic Riccati differential equation [29], stiff systems of ordinary differential equations [30] and nonlinear partial differential equations [31–34], using the perturbation technique.

In the present work, we construct the solution using a different approach. In this work, we obtain an analytical approximation to the solution of the nonlinear Fokker–Planck equation, using a combination of the Laplace Transform and New Homotopy Perturbation Method (LTNHPM). The results obtained via LTNHPM confirm the validity of the proposed method. The rest of this paper is organized as follows:

In Section 2, we illustrate the basic idea behind the new method. In Section 3, the uses of LTNHPM for solving the nonlinear Fokker–Planck equation is presented, and some examples are solved by the proposed method in Section 4. The conclusion appears in Section 5.

2. Analysis of the method

To illustrate the basic ideas behind this method, let us consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (5)$$

with the following initial conditions:

$$u(0) = \alpha_0, \quad u'(0) = \alpha_1, \dots, u^{(n-1)}(0) = \alpha_{n-1}, \quad (6)$$

where A is a general differential operator and $f(r)$ is a known analytical function. The operator, A , can be divided into two parts, L and N , where L is a linear and N is a nonlinear operator. Therefore, Eq. (5) can be rewritten as:

$$L(u) + N(u) - f(r) = 0. \quad (7)$$

By the NHPM [11], we construct a homotopy, $U(r, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$, which satisfies

$$H(U, p) = (1 - p)[L(U) - u_0] + p[A(U) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega. \quad (8)$$

Or, equivalently;

$$H(U, p) = L(U) - u_0 + pu_0 + p[N(U) - f(r)] = 0, \quad (9)$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of the solution to Eq. (5). Clearly, we have, from Eqs. (8) and (9);

$$H(U, 0) = L(U) - u_0 = 0, \quad (10)$$

$$H(U(x), 1) = A(U) - f(r) = 0. \quad (11)$$

By applying the Laplace transform on both sides of Eq. (9), we have:

$$\mathcal{L}\{L(U) - u_0 + pu_0 + p[N(U) - f(r)]\} = 0. \quad (12)$$

Using the differential property of the Laplace transform, we have:

$$\begin{aligned} s^n \mathcal{L}\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \dots - U^{(n-1)}(0) \\ = \mathcal{L}\{u_0 - pu_0 + pN(U) - f(r)\}, \end{aligned} \quad (13)$$

or:

$$\begin{aligned} \mathcal{L}\{U\} = \frac{1}{s^n} \{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0) \\ + \mathcal{L}\{u_0 - pu_0 + p[N(U) - f(r)]\}\}, \end{aligned} \quad (14)$$

By applying the inverse Laplace transform on both sides of Eq. (14), we have:

$$\begin{aligned} U = \mathcal{L}^{-1} \left\{ \frac{1}{s^n} \{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0) \right. \\ \left. + \mathcal{L}\{u_0 - pu_0 + p[N(U) - f(r)]\} \right\}. \end{aligned} \quad (15)$$

According to the HPM, we can first use the embedding parameter, p , as a small parameter, and assume that the solutions of Eq. (15) can be represented as a power series in p , as:

$$U(x) = \sum_{n=0}^{\infty} p^n U_n. \quad (16)$$

Now, let us write Eq. (15) in the following form:

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n \\ = \mathcal{L}^{-1} \left\{ \frac{1}{s^n} \left\{ s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0) \right. \right. \\ \left. \left. + \mathcal{L} \left\{ u_0 - pu_0 + p \left[N \left(\sum_{n=0}^{\infty} p^n U_n \right) - f(r) \right] \right\} \right\} \right\}. \end{aligned} \quad (17)$$

Comparing coefficients of terms with identical powers of p , leads to:

$$\begin{aligned}
p^0 : U_0 &= \mathcal{L}^{-1} \left\{ \frac{1}{s^n} (s^{n-1} U(0) + s^{n-2} U'(0) + \dots \right. \\
&\quad \left. + U^{(n-1)}(0) + \mathcal{L}\{u_0\}) \right\}, \\
p^1 : U_1 &= \mathcal{L}^{-1} \left\{ \frac{1}{s^n} (\mathcal{L}\{N(U_0) - u_0 - f(r)\}) \right\}, \\
p^2 : U_2 &= \mathcal{L}^{-1} \left\{ \frac{1}{s^n} (\mathcal{L}\{N(U_0, U_1)\}) \right\}, \\
p^3 : U_3 &= \mathcal{L}^{-1} \left\{ \frac{1}{s^n} (\mathcal{L}\{N(U_0, U_1, U_2)\}) \right\}, \\
&\vdots \\
p^j : U_j &= \mathcal{L}^{-1} \left\{ \frac{1}{s^n} (\mathcal{L}\{N(U_0, U_1, U_2, \dots, U_{j-1})\}) \right\}, \\
&\vdots
\end{aligned} \tag{18}$$

Suppose that the initial approximation has the form $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1, \dots, U^{(n-1)}(0) = \alpha_{n-1}$, the exact solution may be obtained as follows:

$$u = \lim_{p \rightarrow 1} U = U_0 + U_1 + U_2 + \dots \tag{19}$$

3. LTNHPM applied to the nonlinear Fokker-Planck equation

Consider the nonlinear Fokker-Planck equation in the following form:

$$\frac{\partial u}{\partial t} = - \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, u) + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, u), \tag{20}$$

where $A_i^*(X, t, u) = A_i(X, t, u) u$ and $B_{i,j}^*(X, t, u) = B_{i,j}(X, t, u) u$. For solving this equation by applying the new homotopy perturbation method, we construct the following homotopy:

$$\begin{aligned}
H(U(X, t), p) &= U_t(X, t) - u_0(X, t) + p \left[u_0(X, t) \right. \\
&\quad \left. - \left(- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] = 0,
\end{aligned} \tag{21}$$

where $p \in [0, 1]$ is an embedding parameter, and $u_0(X, t)$ is an initial approximation of the solution to the equation. Clearly, we have, from Eq. (21);

$$H(U(X, t), 0) = U_t(X, t) - u_0(X, t) = 0, \tag{22}$$

$$\begin{aligned}
H(U(X, t), 1) &= U_t(X, t) - \left[- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, u) \right. \\
&\quad \left. + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, u) \right] = 0.
\end{aligned} \tag{23}$$

By applying the Laplace transform on both sides of Eq. (21), we have:

$$\begin{aligned}
&\mathcal{L} \left\{ U_t(X, t) - u_0(X, t) \right. \\
&\quad \left. + p \left[u_0(X, t) - \left(- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) \right. \right. \right. \\
&\quad \left. \left. + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] \right\} = 0.
\end{aligned} \tag{24}$$

Using the differential property of the Laplace transform, we have:

$$\begin{aligned}
&s \mathcal{L}\{U(X, t)\} - U(X, 0) \\
&= \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) - \left(- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) \right. \right. \right. \\
&\quad \left. \left. + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] \right\}.
\end{aligned} \tag{25}$$

Or:

$$\begin{aligned}
\mathcal{L}\{U(X, t)\} &= \frac{1}{s} \left\{ U(X, 0) + \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) \right. \right. \right. \\
&\quad \left. \left. - \left(- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) \right. \right. \right. \\
&\quad \left. \left. + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] \right\}.
\end{aligned} \tag{26}$$

By applying the inverse Laplace transform on both sides of Eq. (26), we have:

$$\begin{aligned}
U(X, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left\{ U(X, 0) + \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) \right. \right. \right. \right. \right. \\
&\quad \left. \left. - \left(- \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) \right. \right. \right. \\
&\quad \left. \left. + \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] \right\} \right\}.
\end{aligned} \tag{27}$$

According to the HPM, we use the embedding parameter, p , as a small parameter and assume that the solutions of Eq. (27) can be represented as a power series in p , as:

$$U(X, t) = \sum_{n=0}^{\infty} p^n U_n(X, t). \tag{28}$$

Substituting Eq. (28) into Eq. (27), and equating the terms with the identical powers of p , leads to calculating $U_j(X, t)$, $j = 0, 1, 2, \dots$

$$\begin{aligned}
p^0 : U_0(X, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} (U(X, 0) + \mathcal{L}\{u_0(X, t)\}) \right\}, \\
p^1 : U_1(X, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ u_0(X, t) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U_0) \right. \right. \right. \\
&\quad \left. \left. \left. - \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U_0) \right\} \right) \right\}, \\
p^2 : U_2(X, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U_0, U_1) \right. \right. \right. \\
&\quad \left. \left. \left. - \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U_0, U_1) \right\} \right) \right\}, \\
&\vdots \\
p^j : U_j(X, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U_0, U_1, \dots, U_{j-1}) \right. \right. \right. \\
&\quad \left. \left. \left. - \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U_0, U_1, \dots, U_{j-1}) \right\} \right) \right\}, \\
&\vdots
\end{aligned} \quad (29)$$

Suppose that the initial approximation has the form $U(x, 0) = u_0(x, t)$, the exact solution may be obtained as follows:

$$u(X, t) = \lim_{p \rightarrow 1} U(X, t) = U_0(X, t) + U_1(X, t) + U_2(X, t) + \dots \quad (30)$$

4. Examples

Example 1. Consider the following nonlinear Fokker–Planck equation taken from [17], such that:

$$\begin{aligned}
u(x, 0) &= x^2, \quad x \in \mathbb{R} \\
A(x, t, u) &= \frac{4}{x}u - \frac{x}{3} \\
B(x, t, u) &= u.
\end{aligned} \quad (31)$$

The exact solution of the above equation was found to be of the form:

$$u(x, t) = x^2 e^t. \quad (32)$$

To solve Eq. (31) by the LTNHPM, we construct the following homotopy:

$$\begin{aligned}
U_t(x, t) - u_0(x, t) + p \left[u_0(x, t) + \frac{8}{x}UU_x - \frac{4}{x^2}U^2 - \frac{U}{3} \right. \\
\left. - \frac{x}{3}U_x - 2U_x^2 - 2UU_{xx} \right] &= 0.
\end{aligned} \quad (33)$$

Applying the Laplace transform on both sides of Eq. (33), we have:

$$\begin{aligned}
\mathcal{L} \left\{ U_t(x, t) - u_0(x, t) + p \left[u_0(x, t) + \frac{8}{x}UU_x - \frac{4}{x^2}U^2 \right. \right. \\
\left. \left. - \frac{U}{3} - \frac{x}{3}U_x - 2U_x^2 - 2UU_{xx} \right] \right\} &= 0.
\end{aligned} \quad (34)$$

Using the differential property of the Laplace transform, we have:

$$\begin{aligned}
s\mathcal{L}\{U(x, t)\} - U(x, 0) &= \mathcal{L} \left\{ u_0(x, t) - p \left[u_0(x, t) \right. \right. \\
&\quad \left. \left. + \frac{8}{x}UU_x - \frac{4}{x^2}U^2 - \frac{U}{3} - \frac{x}{3}U_x - 2U_x^2 - 2UU_{xx} \right] \right\},
\end{aligned} \quad (35)$$

or:

$$\begin{aligned}
\mathcal{L}\{U(x, t)\} &= \frac{1}{s} \left\{ U(x, 0) + \mathcal{L} \left\{ u_0(x, t) - p \left[u_0(x, t) \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{8}{x}UU_x - \frac{4}{x^2}U^2 - \frac{U}{3} - \frac{x}{3}U_x - 2U_x^2 - 2UU_{xx} \right] \right\} \right\}.
\end{aligned} \quad (36)$$

By applying the inverse Laplace transform on both sides of Eq. (36), we have:

$$\begin{aligned}
U(x, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left(U(x, 0) + \mathcal{L} \left\{ u_0(x, t) - p \left[u_0(x, t) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{8}{x}UU_x - \frac{4}{x^2}U^2 - \frac{U}{3} - \frac{x}{3}U_x - 2U_x^2 - 2UU_{xx} \right] \right\} \right) \right\}.
\end{aligned} \quad (37)$$

Suppose the solution of Eq. (37) have the following form:

$$U(x, t) = U_0(x, t) + pU_1(x, t) + p^2U_2(x, t) + \dots \quad (38)$$

where $U_i(x, t)$ are unknown functions which should be determined. Substituting Eq. (38) into Eq. (37), collecting the same powers of p and equating each coefficient of p to zero, results in:

$$\begin{aligned}
p^0 : U_0(x, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left(U(x, 0) + \mathcal{L}\{u_0(x, t)\} \right) \right\}, \\
p^1 : U_1(x, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ u_0(x, t) + \frac{8}{x}U_0U_{0x} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{4}{x^2}U_0^2 - 2U_{0x}^2 - 2U_0U_{0xx} - \frac{U_0}{3} - \frac{x}{3}U_{0x} \right\} \right) \right\}, \\
p^2 : U_2(x, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ \frac{8}{x}U_0U_{1x} + U_1U_{0x} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{4}{x^2}U_0U_1 + U_1U_0 - 2U_{0x}U_{1x} + U_{1x}U_{0x} \right. \right. \right. \\
&\quad \left. \left. \left. - 2U_0U_{1xx} + U_1U_{0xx} - \frac{U_1}{3} - \frac{xU_{1x}}{3} \right\} \right) \right\}, \\
&\vdots
\end{aligned} \quad (39)$$

:

$$\begin{aligned}
p^j : U_j(x, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \left(\mathcal{L} \left\{ \frac{8}{x} \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1x}) \right) \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{4}{x^2} \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1}) \right) \right. \right. \right.
\end{aligned}$$

$$-2 \left(\sum_{k=0}^{j-1} (U_{kx}) (U_{j-k-1x}) \right) - 2 \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1xx}) \right) - \frac{U_{j-1}}{3} - \frac{x(U_{j-1x})}{3} \Bigg\} \Bigg\},$$

∴

Assuming $u_0(x, t) = U(x, 0) = x^2$, and solving the above equation for $U_j(x, t)$, $j = 0, 1, \dots$, leads to the result:

$$U_0(x, t) = x^2(1 + t),$$

$$U_1(x, t) = \frac{x^2}{2}t^2,$$

$$U_2(x, t) = \frac{x^2}{6}t^3,$$

$$U_3(x, t) = \frac{x^2}{24}t^4,$$

$$U_4(x, t) = \frac{x^2}{120}t^5,$$

∴

Therefore, we gain the solution of Eq. (31) as:

$$\begin{aligned} u(x, t) &= U_0(x, t) + U_1(x, t) + U_2(x, t) + \dots \\ &= x^2 \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \dots \right) \\ &= x^2 e^t \end{aligned}$$

which is the exact solution.

Example 2. Consider the following generalized nonlinear Fokker-Planck equation [27], such that:

$$\begin{aligned} u(X, 0) &= x^2, \quad X = (x, y)^t \in \mathbb{R}^2 \\ \begin{cases} A_1(X, t, u) = \frac{4}{x}u \\ A_2(X, t, u) = y, \end{cases} & \begin{cases} B_{1,1}(X, t, u) = u, \\ B_{1,2}(X, t, u) = 1, \\ B_{2,1}(X, t, u) = 1, \\ B_{2,2}(X, t, u) = u. \end{cases} \end{aligned} \quad (41)$$

The exact solution of the above equation was found to be of the form:

$$u(X, t) = x^2 e^t. \quad (42)$$

To solve Eq. (41) by the LTNHPM, we construct the following homotopy:

$$\begin{aligned} U_t(X, t) - u_0(X, t) + p \left[u_0(X, t) + \left(\sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(X, t, U) - \sum_{i,j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}^*(X, t, U) \right) \right] &= 0, \end{aligned} \quad (43)$$

or:

$$\begin{aligned} U_t(X, t) - u_0(X, t) + p \left[u_0(X, t) + \frac{8}{x} U U_x - \frac{4}{x^2} U^2 + U + y U_y - 2 U U_{xx} - 2 U_x^2 - U_{xy} - U_{yx} - 2 U U_{yy} - 2 U_y^2 \right] &= 0. \end{aligned} \quad (44)$$

Applying the Laplace transform on both sides of Eq. (44), we have:

$$\begin{aligned} \mathcal{L} U_t(X, t) - u_0(X, t) + p \left[u_0(X, t) + \frac{8}{x} U U_x - \frac{4}{x^2} U^2 + U + y U_y - 2 U U_{xx} - 2 U_x^2 - U_{xy} - U_{yx} - 2 U U_{yy} - 2 U_y^2 \right] &= 0. \end{aligned} \quad (45)$$

Using the differential property of the Laplace transform, we have:

$$\begin{aligned} s \mathcal{L}\{U(X, t)\} - U(X, 0) &= \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) + \frac{8}{x} U U_x - \frac{4}{x^2} U^2 + U + y U_y - 2 U U_{xx} - 2 U_x^2 - U_{xy} - U_{yx} - 2 U U_{yy} - 2 U_y^2 \right] \right\} \end{aligned} \quad (46)$$

or:

$$\begin{aligned} \mathcal{L}\{U(X, t)\} &= \frac{1}{s} \left\{ U(X, 0) + \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) + \frac{8}{x} U U_x - \frac{4}{x^2} U^2 + U + y U_y - 2 U U_{xx} - 2 U_x^2 - U_{xy} - U_{yx} - 2 U U_{yy} - 2 U_y^2 \right] \right\} \right\}. \end{aligned} \quad (47)$$

By applying the inverse Laplace transform on both sides of Eq. (47), we have:

$$\begin{aligned} U(X, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \left\{ U(X, 0) + \mathcal{L} \left\{ u_0(X, t) - p \left[u_0(X, t) + \frac{8}{x} U U_x - \frac{4}{x^2} U^2 + U + y U_y - 2 U U_{xx} - 2 U_x^2 - U_{xy} - U_{yx} - 2 U U_{yy} - 2 U_y^2 \right] \right\} \right\} \right\}. \end{aligned} \quad (48)$$

Suppose the solution of Eq. (48) to have the following form:

$$U(X, t) = U_0(X, t) + p U_1(X, t) + p^2 U_2(X, t) + \dots, \quad (49)$$

where $U_i(X, t)$ are unknown functions which should be determined. Substituting Eq. (49) into Eq. (48), collecting the same powers of p and equating each coefficient of p to zero, results in:

$$\begin{aligned} p^0 : U_0(X, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s} \{ U(X, 0) + \mathcal{L}\{u_0(X, t)\} \} \right\}, \\ p^1 : U_1(X, t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{s} \mathcal{L} \left\{ u_0(X, t) + \frac{8}{x} U_0 U_{0x} - \frac{4}{x^2} U_0^2 + U_0 + y U_{0y} - 2 U_0 U_{0xx} - 2 U_{0x}^2 - U_{0yx} - U_{0xy} - 2 U_0 U_{0yy} - 2 U_{0y}^2 \right\} \right\}, \end{aligned}$$

$$p^2 : U_j(X, t) = \mathcal{L}^{-1} \left\{ -\frac{1}{s} \mathcal{L} \left\{ \frac{8}{x} U_0 U_{1x} + U_1 U_{0x} - \frac{4}{x^2} U_0 U_1 + U_1 U_0 + U_1 + y U_{1y} - 2 U_0 U_{1xx} + U_1 U_{0xx} - 2 (U_{0x} U_{1x} + (U_{1x}) (U_{0x})) - (U_{1yx}) - (U_{1xy}) - 2 (U_0 (U_{1yy}) + U_1 (U_{0yy})) - 2 ((U_{0y}) (U_{1y}) + (U_{1y}) (U_{0y})) \right\} \right\},$$

⋮

$$p^j : U_j(X, t) = \mathcal{L}^{-1} \left\{ -\frac{1}{s} \mathcal{L} \left\{ \frac{8}{x} \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1x}) \right) - \frac{4}{x^2} \left(\sum_{k=0}^{j-1} U_k U_{j-k-1} \right) + U_{j-1} + y (U_{j-1y}) - 2 \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1xx}) \right) - 2 \left(\sum_{k=0}^{j-1} (U_{kx}) (U_{j-k-1x}) \right) - (U_{j-1yx}) - (U_{j-1xy}) - 2 \left(\sum_{k=0}^{j-1} U_k (U_{j-k-1yy}) \right) - 2 \left(\sum_{k=0}^{j-1} (U_{ky}) (U_{j-k-1y}) \right) \right\} \right\},$$

⋮

Assuming $u_0(X, t) = U(X, 0) = x^2$, and solving the above equation for $U_j(X, t)$, $j = 0, 1, \dots$ leads to the result:

$$\begin{aligned} U_0(X, t) &= x^2 1 + t, \\ U_1(X, t) &= -\frac{x^2}{2} t 4 + t, \\ U_2(X, t) &= \frac{x^2}{6} t^2 6 + t, \\ U_3(X, t) &= -\frac{x^2}{24} t^3 8 + t, \\ U_4(X, t) &= \frac{x^2}{120} t^4 10 + t, \\ &\vdots \end{aligned} \quad (50)$$

Therefore, we gain the solution of Eq. (41) as:

$$\begin{aligned} u(X, t) &= U_0(X, t) + U_1(X, t) + U_3(X, t) + \dots \\ &= x^2 \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \dots \right) \\ &= x^2 e^{-t}, \end{aligned}$$

which is the exact solution.

5. Conclusion

In the present work, we proposed a combination of Laplace transform and homotopy perturbation methods to solve the nonlinear Fokker–Planck equation. The new method, developed in the current paper, was tested on several examples. Unlike the previous approach implemented by the present author, named NHPM, the present technique does not need the initial approximation to be defined as a power series. Also, unlike the well known HPM, there is no need to solve several recurrence differential equations here. The solution approximations can be readily obtained using the inverse Laplace transform. The main advantage of the LTNHPM over ADM is that this method provides the solution without a need for calculating Adomian's polynomials. The obtained results show that these approaches can solve the problem effectively. The computations corresponding to the examples have been performed using Maple 12.

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